

Universal properties of conformal quantum many-body systems

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Abstract

Universal properties of many-body systems in conformal quantum mechanics in arbitrary dimensions are presented. Specially, a general structure of discrete energy spectra and eigenstates is found. Finally, a simple construction of a universal time operator conjugated to a conformal Hermitian or a PT -invariant Hamiltonian is proposed.

Key words: conformal symmetry, quantum many-body systems, time operator

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A number of physical systems exhibit a particular form of asymptotic conformal invariance. There are examples from black holes to molecular and condensed matter physics that can be discussed within a unified treatment. The symmetries relevant here include time translation invariance, scale invariance, and invariance under special conformal transformations that are all part of a larger conformal symmetry with $SO(2, 1)$ group-theory structure [1].

A connection of conformal symmetry with black hole physics has recently been explored in various forms. In particular, the near-horizon symmetry structure of black holes as well as the impact of this symmetry on thermodynamics [2] of black holes has been considered together with its extension to superconformal quantum mechanics [3,4]. In the near-horizon area of black holes, the geometry is that of the three-dimensional anti-de Sitter gravity and, by the Henneaux-Brown conjecture [5], can be described with the help of conformal field theory [6]. By imposing suitable boundary conditions at the horizon, it can be shown that the actual algebra of surface deformations contains a Virasoro algebra in the $(r - t)$ plane. Another route of current applications involves the link of black holes to the Calogero model [7,8], which originates from the fact that the dynamics of particles near the horizon of a black hole is associated with a Hamiltonian containing an inverse square potential [9,10] and conformal symmetry [1]. It is therefore of interest to find quantum states of systems in conformal quantum mechanics, that, in the context of black holes appear to be the horizon states. Some attempts in this direction have been made in various papers [10,11].

In molecular physics, conformal invariance is important in the context of the occurrence of anomaly that is revealed by the failure of symmetry generators to close the algebra. It is shown that the concept of anomaly applies to the inverse square potential, including the electric dipole-charge interaction, with strong implications in molecular physics [12]. Conformal quantum mechanics is also important in condensed matter physics. For example, a correlated two-dimensional N -electron gas with an inverse-square interaction in a magnetic field has been studied within a microscopic analytical theory [13]. In this letter we present simple and universal results for many-body systems in conformal quantum mechanics in arbitrary dimensions.

Let us consider N generally different particles in arbitrary dimensions D , described by a Hermitian or a PT -invariant Hamiltonian [14] of the form

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{1}{m_i} \nabla_i^2 + V(\vec{r}_1, \dots, \vec{r}_N) + \frac{\omega^2}{2} \sum_{i=1}^N m_i \vec{r}_i^2 + \frac{c}{2} \left(\sum_{i=1}^N \vec{r}_i \nabla_i + \frac{ND}{2} \right). \quad (1)$$

The potential V is a real homogeneous function or a PT -invariant operator

of order -2 , i.e., it satisfies the relation

$$[\sum_{i=1}^N \vec{r}_i \nabla_i, V] = -2V. \quad (2)$$

Additionally, we assume that V is invariant under translations, i.e., $[\sum_{i=1}^N \nabla_i, V] = 0$ and, generally, it can be unisotropic [12]. Then the Hamiltonian (1) can be written as $H = -T_- + \omega^2 T_+ + cT_0$, where the generators $\{T_\pm, T_0\}$, defined as

$$\begin{aligned} T_+ &= \frac{1}{2} \sum_{i=1}^N m_i \vec{r}_i^2, \\ T_- &= \frac{1}{2} \sum_{i=1}^N \frac{1}{m_i} \nabla_i^2 - V(\vec{r}_1, \dots, \vec{r}_N), \\ T_0 &= \frac{1}{4} \sum_{i=1}^N (\vec{r}_i \nabla_i + \nabla_i \vec{r}_i) = \frac{1}{2} \sum_{i=1}^N \vec{r}_i \nabla_i + \frac{ND}{4}, \end{aligned} \quad (3)$$

satisfy the $SU(1,1)$ conformal algebra

$$[T_-, T_+] = 2T_0, \quad [T_0, T_\pm] = \pm T_\pm. \quad (4)$$

The generators T_\pm (acting on the Hilbert space of physical states) are Hermitian operators, whereas T_0 is an anti-Hermitian operator. If V is not Hermitian but PT -invariant, so is the T_- generator.

The above system can be viewed as a deformation of N harmonic oscillators in D dimensions, with a common frequency ω . The Hamiltonian (1) can be mapped to $2\omega' T_0$ by transformation:

$$H = 2\omega' S T_0 S^{-1}, \quad (5)$$

where

$$S = e^{-bT_+} e^{-aT_-} \quad (6)$$

and

$$\begin{aligned} a &= \frac{1}{2\omega} \frac{1}{\sqrt{1 + \frac{c^2}{4\omega^2}}}, \\ b &= \omega \left(\sqrt{1 + \frac{c^2}{4\omega^2}} - \frac{c}{2\omega} \right), \\ \omega' &= \omega \sqrt{1 + \frac{c^2}{4\omega^2}}. \end{aligned} \quad (7)$$

If c is real, the new frequency ω' is greater than the original frequency ω , and the Hamiltonian (1) is PT -invariant. If $c = 0$, then $\omega' = \omega$. If c is

pure imaginary, then ω' is less than ω and the cT_0 part, as well as the whole Hamiltonian, is Hermitian. However, for $c = \pm 2i\omega$, it holds $\omega' = 0$, the S transformation becomes singular, and the system is critical. This point represents a boundary between the region of discrete energies which occurs for $\omega'^2 > 0$ and the continuum energy spectrum describing the scattering states that appears for $\omega'^2 < 0$. The case $\omega'^2 < 0$ is related to the notion of inverted oscillator which is relevant to string theory. Note that the norms of the wave functions blow up at the point $\omega' = 0$.

The ground state ψ_0 is well defined if it is a square integrable function. This will be the case if the ground-state energy $\omega'\epsilon_0$, $\epsilon_0 > 0$, is higher than $\omega'\frac{D}{2}$, the condition which is connected with the existence of the critical point [15,16]. Then we can write $\psi_0 = S\Delta_0$, where Δ_0 is a homogeneous function of the lowest degree and of the lowest energy $\omega'\epsilon_0$, satisfying

$$T_-\Delta_0 = 0, \quad T_0\Delta_0 = \frac{\epsilon_0}{2}\Delta_0. \quad (8)$$

There are other homogeneous, generally irrational, solutions Δ_k with homogeneity of higher degrees and with energy $\omega'\epsilon_k = \omega'(\epsilon_0 + k)$, $k > 0$ (generally, there are degenerate states with the same k). They satisfy

$$T_-\Delta_k = 0, \quad T_0\Delta_k = \frac{k + \epsilon_0}{2}\Delta_k, \quad k > 0. \quad (9)$$

Since there is a class of PT -invariant Hamiltonians that have real energy spectra [14], we are free to restrict ourselves to the cases where the energies $\omega'\epsilon_k$ of the system (1) are real. For example, if $V = 0$, then Δ_k are harmonic polynomials in ND dimensions with energies $\omega'(\frac{ND}{2} + k)$, where $k = 0, 1, 2, \dots$. The construction of solutions of Eqs. (9) with $V \neq 0$, is generally a difficult task.

The universal set of excited states is $\psi_{n,k} = ST_+^n\Delta_k$, $n = 0, 1, 2, \dots$; $k \geq 0$, with energies $2\omega'(n + \frac{\epsilon_k}{2})$. For a given $k \geq 0$, the spectrum is equidistant with an elementary step $2\omega'$. Hence, all solutions are grouped into equidistant towers based on $S\Delta_k$, $k \geq 0$. Specially, for identical particles in one dimension, the states $\psi_{n,k} = ST_+^n\Delta_k$, $n, k = 0, 1, 2, \dots$, represent the complete set of S_N symmetric solutions, with an equidistant energy spectrum with the step ω' .

Note that there is a universal radial equation for a radial part $\phi_{n,k}(T_+)$ of the wave function $\psi_{n,k} = \phi_{n,k}(T_+)\Delta_k$. Using the radial representation of the generators T_{\pm}, T_0 , with T_+ as a radial variable, Eq.(3), and with

$$T_- = T_+ \frac{d^2}{dT_+^2} + \epsilon_k \frac{d}{dT_+}, \quad T_0 = T_+ \frac{d}{dT_+} + \frac{\epsilon_k}{2}, \quad (10)$$

then by applying them to the $S\Delta_k$ tower, one obtains the universal radial equation

$$[T_+ \frac{d^2}{dT_+^2} + (\epsilon_k - cT_+) \frac{d}{dT_+} + (E_{n,k} - \omega^2 T_+ - c \frac{\epsilon_k}{2})] \phi_{n,k}(T_+) = 0. \quad (11)$$

It follows directly upon substituting the factorization $\psi_{n,k} = \phi_{n,k}(T_+) \Delta_k$ into the eigenvalue equation $H\psi_{n,k} = E_{n,k}\psi_{n,k}$, with solutions $\psi_{n,k} = ST_+^n \Delta_k$, $n = 0, 1, 2, \dots$ and with the corresponding energies $E_{n,k} = \omega'(2n + \epsilon_k)$. The solution of Eq. (11) is of the form $\phi_{n,k}(T_+) = F_{n,k}(T_+) e^{-\omega' T_+}$, where, $F_{n,k}(T_+)$ is an associated Laguerre polynomial $L_{n+\epsilon_k-1}^{\epsilon_k-1}(2\omega T_+)$ for $c = 0$. The set of operators $\{T_\pm, T_0\}$ can be separated into the center-of-mass and relative parts: $T_\pm = (T_{CM})_\pm + (T_{rel})_\pm$ and $T_0 = (T_{CM})_0 + (T_{rel})_0$ with the same separation for the energy parameter ϵ_k . The universal radial equation then splits into two equations of the form (11), namely, for $(T_{CM})_+$ and for $(T_{rel})_+$. Generally, the center-of-mass (CM) motion is described by the D -dimensional oscillator, see Refs. [15,16]. Then the excited states can be written as $\psi_{n_\alpha, n, k} = S \prod_{\alpha=1}^D (R_\alpha)^{n_\alpha} (T_{rel})_+^n \Delta_k$, where $\vec{R} = \frac{\sum_{i=1}^N m_i \vec{r}_i}{\sum_{i=1}^N m_i}$ is the center-of-mass vector.

For example, for an arbitrary conformal quantum many-body system in two dimensions with identical particles [8,13,16], the corresponding states are

$$S(\sum_{i=1}^N z_i)^l (\sum_{i=1}^N \bar{z}_i)^{\bar{l}} (T_{rel})_+^n \Delta_k, \quad (12)$$

where z_i, \bar{z}_i are complex coordinates. These states are the eigenstates of the angular momentum operator, with the eigenvalue $L_{CM} = l - \bar{l}$ due to the CM motion.

The Fock space corresponding to $\psi_{n,k} = ST_+^n \Delta_k$, $n = 0, 1, 2, \dots$; $k \geq 0$, is spanned by $\prod_{\alpha=1}^D (A_{1,\alpha}^+)^{n_\alpha} (B_2^+)^n \psi_{0,k}$, where $\psi_{0,k} = S\Delta_k$ are vacua of corresponding towers. We have introduced the operators $A_{1,\alpha}^\pm$ and B_2^\pm defined by

$$\begin{aligned} A_{1,\alpha}^+ &= S R_\alpha S^{-1}, & A_{1,\alpha}^- &= S \nabla_\alpha S^{-1}, \\ B_2^\pm &= S (T_{rel})_\pm S^{-1}. \end{aligned} \quad (13)$$

Furthermore, we construct universal phase-angle variables and a time operator conjugated to the Hamiltonian (1), generalizing the result of Ref. [17]. From

the $SU(1, 1)$ algebra it follows

$$\begin{aligned} T_+ T_- &= \phi(T_0) = (T_0 - \frac{\epsilon}{2})(T_0 + \frac{\epsilon}{2} - 1), \\ T_- T_+ &= \phi(T_0 + 1) = (T_0 - \frac{\epsilon}{2} + 1)(T_0 + \frac{\epsilon}{2}), \end{aligned} \quad (14)$$

where $\omega'\epsilon$ is the generic energy of the lowest state in the corresponding tower (cf. Eq. (9)). Also, for an arbitrary function $f(T_0)$ which has the power series expansion, it holds $T_- f(T_0) = f(T_0 + 1)T_-$ and $T_+ f(T_0) = f(T_0 - 1)T_+$. The Casimir operator is $-T_+ T_- + T_0(T_0 - 1)$ with the eigenvalue $\frac{\epsilon_k}{2}(\frac{\epsilon_k}{2} - 1)$ when acting on the particular tower based on Δ_k . The quantity $\frac{\epsilon_k}{2} > 0$ is a spin of a discrete irreducible representation of the universal covering group [18].

Now we can introduce the operator

$$Q = T_+ \frac{\imath}{T_0 + \frac{\epsilon}{2}} \quad (15)$$

which is conjugated to $-T_-$ for each tower of states built on Δ_k (for $\epsilon > 0$, the energy spectrum is discrete). Then the following relations hold:

$$[Q, -T_-] = \imath, \quad [Q, T_+] = \imath Q^2, \quad [Q, T_0] = -Q. \quad (16)$$

It is easy to verify that the adjoint form of the relations (16) looks like

$$[Q^\dagger, -T_-] = \imath, \quad [Q^\dagger, T_+] = \imath Q^{\dagger 2}, \quad [Q^\dagger, T_0] = -Q^\dagger.$$

Since Q is not a Hermitian operator, we define a Hermitian one as $\mathcal{T}_0 = \frac{1}{2}(Q + Q^\dagger)$. Now the operator conjugate to the Hamiltonian $H = -T_- + \omega^2 T_+ + cT_0$ is obtained by solving the equation

$$[\mathcal{F}(Q), H] = \imath. \quad (17)$$

Anticipating the adjoint form of relations (16), note that this relation directly implies $[\mathcal{F}(Q^\dagger), H] = \imath$. By using Eqs. (16), the relation (17) is reduced to

$$\frac{d\mathcal{F}}{dQ}(1 + \omega^2 Q^2 + \imath cQ) = 1$$

and the result for the time operator \mathcal{T} is

$$\mathcal{T} = \frac{1}{2}(\mathcal{F}(Q) + \mathcal{F}(Q^\dagger)), \quad (18)$$

where

$$\mathcal{F}(Q) = \frac{1}{\sqrt{1 + \frac{c^2}{4\omega^2}}} \frac{1}{2\omega} \text{arctg} \left(\frac{\omega Q + \frac{ic}{2\omega}}{\sqrt{1 + \frac{c^2}{4\omega^2}}} \right). \quad (19)$$

In the case that the Hamiltonian H is Hermitian, the time operator \mathcal{T} , Eq. (18), is also Hermitian.

The time operator can be written in terms of the logarithmic function by using the identity $\text{arctg} x = \frac{1}{2i} \ln \frac{1+ix}{1-ix}$. There are generally many time operators corresponding to towers (generally, there are an infinite number of towers) with different $\epsilon_k = \epsilon_0 + k$, $k \geq 0$. Besides this, one can add an arbitrary function of the Hamiltonian to \mathcal{T} , Eq.(18), without changing anything. One can also define the time operator in respect to the center-of-mass and relative Hamiltonians. Our result, Eq.(18), for the time operator is a simple universal property of all systems with the underlying $SU(1, 1)$ symmetry.

In the case of one oscillator in D dimensions ($V = 0$), the operator Q , Eq.(15), reduces to the form (in the following we set $m = 1$).

$$Q = \vec{R}^2 (\vec{P} \cdot \vec{R})^{-1}, \quad \vec{P} = -i\nabla. \quad (20)$$

Particularly, the result (18) is a generalization of Ref. [17]. For one harmonic oscillator in one dimension with vanishing potential ($V = 0$), we obtain

$$\mathcal{T} = \frac{1}{2\omega} \text{arctg} \left(\omega x \frac{1}{p} \right) + h.c., \quad (21)$$

coinciding with Ref. [17]. In the limiting case $c = 0$, $\omega \rightarrow 0$, we obtain $\mathcal{T} = \mathcal{T}_0 = \frac{1}{2}(Q + Q^\dagger)$ and for $V = 0$, it reduces to the time operator of the free particle

$$\mathcal{T} = \frac{1}{2} \left(x \frac{1}{p} + \frac{1}{p} x \right). \quad (22)$$

This quantity coincides exactly with the time-of-arrival operator of Aharonov and Bohm [19].

Finally, one can introduce the operators

$$A_2^\pm = S(2\omega')^{\pm 1} T_\pm S^{-1} = \frac{(\omega' \pm \frac{c}{2})^2}{2\omega'} T_+ + \frac{1}{2\omega'} T_- - \left(\frac{c}{2\omega'} \pm 1 \right) T_0, \quad (23)$$

where ω' is given in the relation (7) and c is the parameter appearing in

front of the PT -invariant term in the expression (1). Owing to the relations $[H, A_2^\pm] = \pm 2\omega' A_2^\pm$, the time operator can be written as

$$\mathcal{T} = -\frac{\imath}{4\omega'} \left(\ln A_2^+ - \ln A_2^- \right). \quad (24)$$

Note that the operator $\tilde{\mathcal{T}} = -\frac{\imath}{4\omega'} \left(\ln T_+ - \ln T_- \right)$ is conjugated to $2\omega' T_0$, i.e., $[\tilde{\mathcal{T}}, 2\omega' T_0] = \imath$, where $\tilde{\mathcal{T}} = S^{-1} \mathcal{T} S$, and S is the rotation operator defined by (6). The coherent states can be constructed similarly as in Refs. [17,18].

In summary, we have presented some simple and universal properties of a general class of many-body systems in conformal quantum mechanics in arbitrary dimensions. Particularly, the radial equation, common to all systems with underlying conformal symmetry, has been constructed, and a general structure of discrete energy spectra and eigenstates has been found. The universal form for the operator conjugated to the Hamiltonian has also been found. The methods presented here can be extended to N -body systems in superconformal quantum mechanics.

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